Change in Distribution of Latent Ability with Item Position in a CAT Sequence

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Introduction

Very often definition of item parameters for item used in Computer Adaptive Test (CAT) is done in Paper and Pencil (P&P) tests, where examinees ability are distributed approximately normal (Segal et al, 1997). However in CAT environment ability distribution of examinees, who are getting the given item in the test, can differ considerably from "original" close to normal distribution. The nature of this phenomena and it possible influences on item performance in the CAT testing is discussed in the paper.

In this paper, we will use experimental material from the Arithmetic Reasoning (AR) test of the Armed Services Vocational Aptitude Battery (ASVAB). About 500,000 examinees take the ASVAB each year, so we have a rather broad ability distribution of examinees. We have chosen the AR test from the ASVAB because some studies show that the AR test is uni-dimensional (Zimowsky & Bock, 1987), which elevates many complications in the our study.

1. Some Facts from ASVAB Testing

Due to the adaptive nature of a CAT test, the posterior ability distribution of a group of examinees who get a particular item from a CAT item pool differs drastically from the original ability distribution. This change in posterior ability distribution can influence the functioning of following items that were calibrated using the original distribution, which is typically close to normal $\mathrm{N}(0,1)$. Keep in mind that the posterior ability for the n-th position is the prior distribution for the item given in the $(n+1)$-th position.
The CAT-ASVAB tests are adaptive, in which the selection of an item is driven by the maximum information for the particular range of ability. Note, however, that the item selection process is also subject to the Sympson-Hetter exposure-control mechanism (Sympson & Hetter, 1985). All items in the test are assumed to follow the 3PL model, i.e., any item in the CAT-ASVAB pool is totally characterized by its discriminating, difficulty, and guessing parameters (a_i, b_i, c_i), $i = 1, \dots, M$, where M is the total number of items in the CAT adaptive pool.

To estimate the change in posterior ability distribution with item position in the CAT sequence, we use data collected in the last two years from about 300,000 examinees. As an ability estimator for a particular examinee, we use his/her CAT ability estimate (CAT score) which is made by successive applications of Bayesian-Owen and Bayesian-Modal algorithms.
Table1 shows the overall usage of selected items from the AR test of Form 1 of the CAT-ASVAB. Altogether, the AR-CAT1 pool of adaptive items is 94.

Figure 1 shows the change in prior (posterior for previous step) ability distribution for a few items in the AR test. The left graph shows the change in prior ability distribution for a group of examinees that got items of "average" difficulty in position 1 (item AR0144 with parameters $a = 1.57$, $b = -0.35$, $c = 0.11$) and position 8 (item AR0171 with parameters: $a = 1.64$, $b = 0.79$, $c = 0.06$). The right graph shows ability distributions for two items in the same position 11 (an easy item, AR0061 with parameters $a = 1.05$, $b = -1.81$, $c = 0.18$; and a hard item, AR0391, with parameters $a = 1.77$, $b = 1.26$, $c = 0.25$).

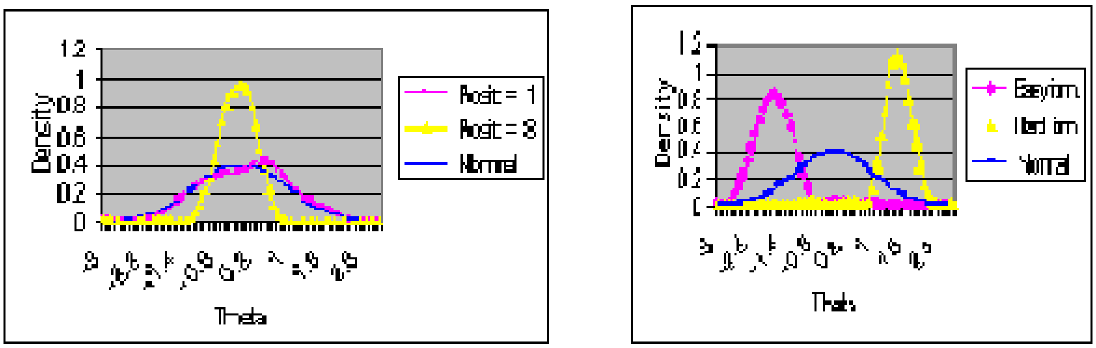

Figure 1. Change in ability distribution for a group of examinees, who got the given items in different positions of a CAT sequence.

For the density estimation of ability we use at least 1,200 examinees who get the correspondent item in the indicated position of their regular AR-CAT ASVAB test. For the smoothed approximation of density we use the ForScore algorithm by Dr. M. Levine and B. Williams (1998). Results show the positive domain of density of the ability distribution contracts as the test progresses, and at position 8 or higher it usually has a length of not more than 0.3 on the standard CAT-ASVAB scale (-2.5, 2.5).

The described above phenomenon can be traced theoretically. The value of examinee ability $q \in [-2.5, 2.5]$ can be approximated by maximizing value of examinee likelihood:

$$L(\underline{u}, q) = g(q) \cdot \prod_{k=1}^{K} P_{i_k}(q)^{u_k} \cdot Q_{i_k}(q)^{(1-u_k)},$$

where $\bar{u} = \{u_1, u_2, \cdots, u_K\}$ is the sequence of binary answers for an examinee taking a CAT test, i_k is the item number reached by the examinee in position number k, and K is the test length ($K = 15$ in the case AR test). Here $P_i(q) = c_i + \dfrac{1 - c_i}{1 + \exp(l_i(q))}$, $l_i(q) = -D \cdot a_i \cdot (q - b_i)$ is 3PL Item Characteristic Curve (ICC) for the item i, and $Q_i(q) = 1 - P_i(q)$, and $g(q)$ is the density of prior distribution, assumed to be normally distributed: $N(0,1)$. Let's assume that the examinee reaches item i_j in position $j < K$. It can be shown that under some no-divergency conditions (formulated further) and in absence of exposure control mechanism there is a unique path $\{(\bar{i}_k, \bar{u}_k)\}_{k=1}^{k=j}$, of items and answers to them which will force an examinee to have item i_j as item number j in his/her CAT test sequence. Then we can rewrite examinee likelihood in the form:

$$L(\underline{u}, q) = C_j \cdot G_j(q) \cdot \prod_{k=j}^{K} P_{i_k}(q)^{u_k} \cdot Q_{i_k}(q)^{(1-u_k)},$$

where $G_j(q) = g(q) \cdot \prod_{k=1}^{j-1} P_{\bar{i}_k}(q)^{u_k} \cdot Q_{\bar{i}_k}(q)^{(1-u_k)} \cdot \frac{1}{C_j}$, and C_j is the normalizing constant, i. e., $\int_{-2.5}^{2.5} G_j(q)dq = 1$. Then $G_j(q)$ can be considered as the ability distribution for an examinee who gets item i_j in the j-th position of the CAT test.

For example, for item AR0171, described earlier, to be reached on position # 8 a "typical" path of item and answers is shown in the Table 1:

Position 1:	AR9929	0	(2.11, -0.05, 0.18)
Position 2:	AR0031	1	(1.47, -0.61, 0.13)
Position 3:	AR0004	1	(1.60, -0.33, 0.10)
Position 4:	AR9822	1	(1.79, -0.22, 0.22)
Position 5:	AR9007	1	(2.10, 0.16, 0.17)
Position 6:	AR1213	0	(2.19, 0.46, 0.12)
Position 7:	AR1305	1	(1.91, 0.31, 0.10)
Position 8:	AR0171	0	(1.64, 0.79, 0.06)

Table 1. Example of items chain in AR ASVAB-CAT1 test.

It is easy to see that due to transformation:

$$G_j(q) = g(q) \cdot \prod_{k=1}^{j-1} P_{\bar{i}_k}(q)^{u_k} \cdot Q_{\bar{i}_k}(q)^{(1-u_k)} \cdot \frac{1}{C}$$

of original prior normal distribution to the intermediate distribution which is prior for the item i_j in the sequence of the CAT test, any incorrect answer "cut" a little bit the right tail of normal distribution, and any correct answer "cut" it left tail.

Figure 2 presents the results of the "application" of an incorrect answer in the first element of the sequence presented above on the normal ability density. As we can see, the result of the product $(1 - P_1(q)) \cdot g(q)$ decreased the right tail of the normal distribution, due to the exponential decreasing of the function $(1 - P_1(q))$. On the other hand, the result of the product $P_1(q) \cdot g(q)$ in the case of the correct answer on the first item will cut the left tail of prior distribution. Under additional constraints, the distribution $G_j(q)$ is unimodal, with a contracting domain of its positive values when j is increasing.

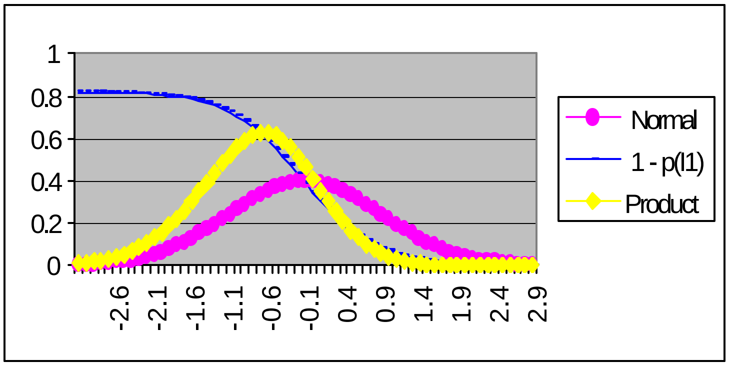

Figure 2. Result of first wrong answer on distribution $G_j(q)$.

As we already mentioned any randomization due to exposure control or imprecision in computing of examinee ability distribution will destroy of the above uniqueness of CAT test sequence. Due to that, as it can be seen in the Table 2.1 and Table 2.2, the majority of examinees got a particular item beginning from the curtain position of their test sequence although there are small number of examinees who get this item on previous positions. This "smoothing" of the rigid "barrier" allocation pattern is due mostly to randomization caused by exposure control in CAT ASVAB. Examples in the tables 2.2 and 2.2 shows that the phenomena for the item to be first time "administered" to the examinee on the "late" position in the CAT test can appear for more or less "normal" items. For example, item AR0171 which is not very hard item first time administered to the group of examinees large enough for calibration beginning from the position 8. In this case we have some administration of the item on previous positions 6 and 7, probably due to item exposure mechanism.

Pos 1	Pos 2	Pos3	Pos4	Pos5	Pos 6	Pos7	Pos 8	Pos 9	Pos10	Pos11	Pos12	Pos13	Pos14	Pos15
00000	04599	00000	17422	16971	16449	08001	10196	06209	05283	04396	04852	03352	02832	02532
00000	03169	00242	16836	19633	13209	09538	09921	05857	05214	04670	03712	02644	02749	02067
00000	00000	00000	00000	00000	00000	00000	00000	00000	00034	00516	00792	00865	01295	01323
13280	22904	30236	03203	07172	02120	01459	01629	00749	00609	00690	00340	00295	00512	00405
00000	00000	00000	00000	00000	00037	00311	01871	05031	10910	11124	12760	10791	09281	09310
00000	00000	00000	00000	00000	00000	00000	00000	00000	00000	03397	03399	04602	04159	06319

Table 2.1. Example of factual item usage in AR ASVAB-CAT1 test.

AR0032 a= 2.180, b= 0.410, c= 0.280
AR0031 a= 1.470, b= -0.610, c= 0.130
AR0061 a= 1.050, b= - 1.810, c= 0.180
AR0144 a= 1.570, b= - 0.350, c= 0.110
AR0171 a= 1.640, b= 0.790, c= 0.060
AR0391 a= 1.770, b= 1.260, c= 0.250

Table 2.2. Items from AR CAT1 pool correspondent to the Table 1.1.

The uniqueness of the sequence in the CAT test is closely connected with uniqueness of CAT "solution" in the sense of the CAT sequence for the examinee with the given ability. In the Appendix we will explore condition of the uniqueness in the case of Bayesian approach in computing of the next best item for an examinee with the given ability (which is used in CAT-ASVAB) as well as for the case of maximum likelihood "on fly" estimation of examinee ability (which with special fast maximizing algorithms can be used in CAT environment).

2. Recalibration of Adaptive Items.

The ability to recalibrate CAT adaptive item is very important in CAT testing environment because with time some items began to be compromised and some items are losing original sense, especially in technical tests, due to technology change.

The CAT ASVAB has a built in mechanism for calibration of new items – the seeded item scheme (Segal et al, 1997). In the seeded item scheme, a new item (candidate for calibration) is given to every examinee in the second, third or fourth position of each test. The item is chosen randomly from the set of seeded items (currently the set contains 100 new items per test). With this approach we collect answers for a new item in operational environment, which increase precision of testing. Our calibration packages require to have at least 1,200 answers for a new item in order to obtain statistically stable estimation of its parameters (Krass, 1998).

To recalibrate CAT adaptive items we developed a pseudo-seeded item scheme. If in the set of studied CAT tests (usually we have about 500,000 CAT tests), we found that the particular CAT adaptive item used more than 1,200 times in the same position for the different examinees. We took the correspondent subset of tests as the set of tests in pseudo-seeded item approach for the given adaptive item. We consider in this case the adaptive item as a seeded item and eliminate from the test position on which the adaptive item was applied artificially decreasing the length of the test. For example in Table 1 the item AR0171 can be considered as pseudo seeded item on positions 8 through 15, and length of the test will be 14 instead of 15 for the real AR test. This approach is giving possibility of recalibrate of great majority of CAT adaptive items and study influence of recalibration parameters from position of the item in CAT test sequence.

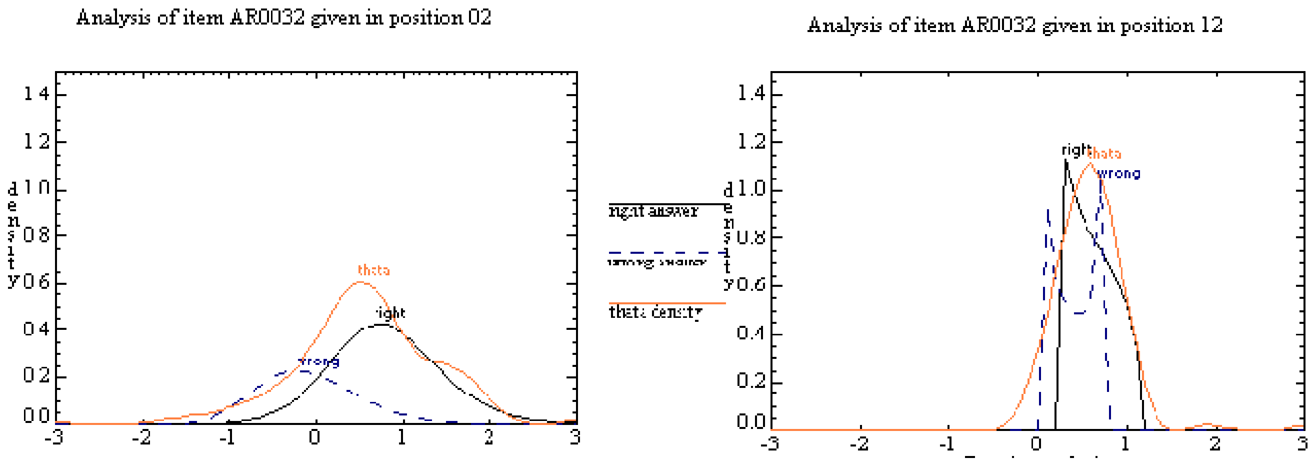

Figure 3. Changing of prior distribution depending on test position.

As we described earlier in this paper, there is a drastic change in the prior ability distribution for the examinees who took the specific item in the given position of their CAT; i. e., ability distribution is conditional on position number and the adaptive test item in this position. In Figure 3 we present the prior distribution for the same item in positions 2 and 12. We also show the ability distributions for examinees who chose correct or incorrect answers to the item estimated by the Pearson differential equation curve (saving four first moments of experimental distribution). As we can see, if the item is in position 2, the distribution of ability is close to normal with a shifted mean (this is usual for the AR test). However, if the item is in position 12, the distribution is very compressed.
After recalibration we found that standard error in the estimation of item difficulty can increase up to 5.5 times in the extreme case when the same item is used in the first position and 15 positions. The same effect even more aggravated is in the case of discrimination and guessing parameters. This phenomenon is due to strongly distortion in prior ability distribution. Thus recalibration of CAT adaptive items (CAT item pool) should be done very cautiously with preliminary estimation position on which examinee in the estimated group began to get the item first time in their CAT sequence. From the other point of view we did not find influence on item difficulty, except increasing SE in evaluation procedure, the increasing in the position number in which the item is evaluated. This influence can be due to fatigue effect of an examinee and should lead to increasing item difficulty when the test is progressing.

Appendix: About uniqueness of solution in CAT test.

1. Baysian Approach.

The CAT-ASVAB uses Bayesian approach to estimate ability of an examinee q_i on the step $i = 1,\ldots,K$ of CAT test. If an examinee estimated ability was q_{i-1} on the previous step and his/her binary response is u_i for item $g_{I(i)} = (a_{I(i)}, b_{I(i)}, c_{I(i)})$ which the examinee got from CAT selection algorithm; then, due to J. Owen (1975), examinee Bayesian estimation of q_i can be presented in the form:

$$q_i = q_{i-1} + w_{i-1} \cdot R_i \cdot (\frac{u_i}{A_i} - 1). \quad (1)$$

Here w_i is variance of ability distribution of examinees who used sequence of binary answers $\{u_i\}_{i=1}^{i=j}$ from the beginning of CAT test. We are using CAT-ASVAB assumption that initial estimation of any examinee ability $q_0 = 0$. Other variables in the formula (1) are:

$R_i = \frac{g(d_i)}{G(d_i) \cdot \sqrt{a_{I(i)}^{-2} + w_{i-1}}}$, $A_i = c_{I(i)} + (1 - c_{I(i)}) \cdot G(-d_i)$ and $d_i = \frac{b_{I(i)} - q_{i-1}}{\sqrt{a_{I(i)}^{-2} + w_{i-1}}}$, where $g(q)$ is a density of normal $N(0,1)$ ability distribution and $G(q)$ its accumulative distribution. The variance w_i can be expressed as function of $u_i, g_{I(i)}, w_{i-1}$ which due to briefness we did not present. However, due to Bayesian character of transformation (1) we always have $w_i \le w_{i-1}, \quad i = 1, \ldots, K$, where $w_0 = 1$.

Getting ability estimation q_i the CAT selection algorithm will chose next item $I(i) = \bar{S}(q_i)$, which using (1) can be presented as:

$$I(i) = S(u_i, q_{i-1}), \quad (2)$$

where S is transformation of selection algorithm $\bar{S}(q_i)$ after substitution from (1).

Let $B(\Theta)$ denote a Bayesian extension of original ability domain $\Theta = [-2.5, 2.5]$

$$B(\Theta) = \bigcup_{q \in \Theta, u \in \{0,1\}, w \le w_0} \Omega(q, u, w),$$

where $\Omega(q, u, w)$ is CAT Bayesian transformation defines by (1). In the other words $B(\Theta)$ is domain of all feasible q which can be reached by CAT algorithm with given CAT item pool, beginning from normal distribution $G(q)$.

Definition: We will call a CAT rest is **not divergent** if

$$S(u, q) \ne S(\bar{u}, q), \quad (3)$$

for any $q \in B(\Theta)$ and any complimentary binary responses $u, \bar{u}$, such that $u + \bar{u} = 1$.

In the other words (3) means that if an examinee in any point of the test will change his binary response (this experiment can be done only imaginative), he/she will get another item for the next step of CAT test. If CAT satisfies non-divergency assumption it means that its item pool is rich enough and selection algorithm (2) is flexible enough.

Further we assume that 3PL parameters for any item from the CAT bank $(a_i, b_i, c_i), \; i = 1, \ldots, M$ are just statistical estimation of its "true" or latent parameters, which is done by process of calibration. Thus, the item parameters for any CAT pool items are particular representation of random variable which is product of used calibrating tool. In the case of CAT ASVAB the tool is Bilog for CAT1-CAT4 item pool or On-line calibration algorithms (see Krass, 1998) in the current Seeded item environment.

Proposition 1. Let the prior examinee ability distribution be normal $N(0,1)$ with initial ability estimation of any examinee $q_0 = 0$. Let the CAT algorithm uses the Bayesian update, the CAT be non-divergent, and there is no randomization of exposure control mechanism in the item selection algorithm. Then, for every item $I(j)$ which can be reached by an examinee on the step $j < K$, there is a unique binary sequence $\{u_k\}_{k=1}^{k=j-1}$ of answers, which ensures that the examinee will get the item $I(j)$ on his/her j - th sequence of the CAT test.

Proof.
Recursively using (1) and (2) we can write

$$I(j) = \Psi(u_1,\ldots,u_{j-1};i_0,\ldots,i_{j-1}) = i_j\,, \tag{4}$$

where $\{u_k\}_{k=1}^{k=j-1}$ is the sequence of binary answers and $\{i_k\}_{k=0}^{k=j-1}$ is the correspondent sequence of items which provide examinee to get item $I(j)$ on his/her j -th position in the CAT test.

Now, let's assume that contrary to theorem statement, there is another sequence of binary answers

$$\{\tilde{u}_k\}_{k=1}^{k=j-1} \neq \{u_k\}_{k=1}^{k=j-1} \tag{5}$$

and sequence of selected items $\{\tilde{i}_k\}_{k=0}^{k=j-1}$ such that:

$$I(j) = \Psi(\tilde{u}_1,\ldots,\tilde{u}_j;\tilde{i}_0,\ldots,\tilde{i}_{j-1}) = \Psi(u_1,\ldots,u_j;i_1,\ldots,i_{j-1})\,. \tag{6}$$

Due to the non-divergency assumption and (5) we have $\{\tilde{i}_k\}_{k=0}^{k=j-1} \neq \{i_k\}_{k=0}^{k=j-1}$, Then, utilizing the Baysian transformation (1) and the selection algorithm, we can see that (6) presents a non-trivial connection between item coefficients, which can be presented in the form:

$$\Phi((\tilde{a}_1,\tilde{b}_1,\tilde{c}_1),\ldots,(\tilde{a}_j,\tilde{b}_j,\tilde{c}_j),(a_1,b_1,c_1),,\ldots,(a_j,b_j,c_j)) = 0\,, \tag{7}$$

where $\Phi(\cdot)$ is a smooth function defined on $R^{3\cdot j}$ Euclidean space. Then, if at least one of the derivatives

$$\frac{\partial\Phi(x_1,\ldots,x_{3j})}{\partial x_q} \neq 0;\ q = 1,\ldots,3\cdot j\ , \tag{8}$$

equation (7) is defined at least $(3*j-1)$ dimensional manifold in $R^{3\cdot j}$ (see, for example, Kantorovich, 1965). Here $x_q,\ q = 1,\cdots,3\cdot j$ is the formal variable of the function Φ. Because, as we assumed, all 3PL coefficients present a statistical estimation of "true" 3PL coefficient probability of holding, (6) is zero.

If (8) is not holding, then assuming infinite smoothness of all functions in consideration we will see that 3PL coefficients of items which forms sequences $\{\tilde{i}_k\}_{k=0}^{k=j-1}$ and $\{i_k\}_{k=0}^{k=j-1}$ belong to another $(3*j-k),\quad k>1$ manifold, which also contradicts latency assumption in the definition of 3PL parameters of adaptive items.

Corollary.
If CAT test is not divergent then there is one-to-one correspondence between binary response sequence $\{u_i\}_{i=1}^{i=K}$ and CAT ability estimation q_K of examinee, who used this response sequence.
This corollary is consequence of uniqueness of Baysian transformation (1) and item selection algorithm (2) in the case of absence of exposure control.

2. Maximum Likelihood Approach.

We are assume that prior distribution of examinees ability is normal $g(q) = \frac{1}{\sqrt{2p}} \cdot e^{-\frac{q^2}{2}}$, where $q \in [q_{\min}, q_{\max}]$. In cat ASVAB $-q_{\min} = q_{\max} = 2.5$. For item $i = 1, \ldots, M$ with 3PL parameters (a_i, b_i, c_i) its ICC is: $p_i(q) = c_i + \frac{1 - c_i}{1 + e^{-Da_i(q - b_i)}}$ and $q_i(q) = \frac{1 - c_i}{1 + e^{Da_i(q - b_i)}}$.

From that we got

$$g'(q) = -q \cdot g(q), \qquad (9)$$

and

$$q_i'(q) = D \cdot a_i \cdot q_i(q) \cdot e_i(q), \qquad (10)$$

where $e_i(q) = -\frac{e^{Da_i(q - b_i)}}{1 + e^{Da_i(q - b_i)}}$.On the Figure 4 we presented the function $e_i(q)$ for item AR0004 with parameters: (1.60, -0.33, 0.10).

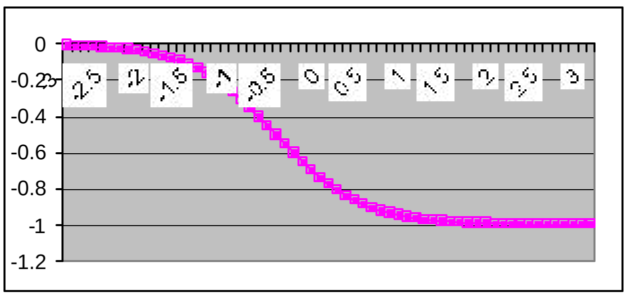

Figure 4. Example of the function $e_i(q)$ (item AR0031).

As we can see the function is monotone decreasing with most negative derivative in the value $\hat{q} = b_i$. Also

$$p_i'(q) = -q_i'(q) = p_i(q) \cdot (-D \cdot a_i \cdot \frac{q_i(q)}{p_i(q)} \cdot l_i(q)) = D \cdot a_i \cdot p_i(q) \cdot E_i(q), \qquad (11)$$

where $E_i(q) = \frac{(1 - c_i) \cdot e^{Da_i(q - b_i)}}{(c_i + e^{Da_i(q - b_i)}) \cdot (1 + e^{Da_i(q - b_i)})}$. The function $E_i(q)$ is depicted on the Figure 5 for the same AR item.

The function $E_i(q)$ is unimodal and is reaching maximum at the point $e^{Da_i(\bar{q}_{\max} - b_i)} = \sqrt{c_i}$ or $\bar{q}_{\max} = \frac{1}{2D \cdot a_i} \ln c_i + b_i = -1.106$. If guessing parameter c_i small enough maximizing point $\bar{q}_{\max}$ is lesser than left boundary point $q_{\min}$ which mean that in this case the function $E_i(q)$ is monotone decreasing in the feasible area $[q_{\min}, q_{\max}]$ or its derivative is lesser than zero. In the considered case of item AR0031 the function $E_i(q)$ is monotone decreasing in the are of positive abilities. Out of the formula for $\overline{q}_{\max}$ it is followed that the function $E_i(q)$ is monotone decreasing in the point $q = b_i$ because $0 \le c_i < 1$, even $c_i \le 0.25$ in the case of AR test. Maximum value of this function $E_i(\overline{q}_{\max}) = D \cdot a_i \cdot \frac{1-\sqrt{c_i}}{1+\sqrt{c_i}} = 0.97$ lesser than 1 for considered item.

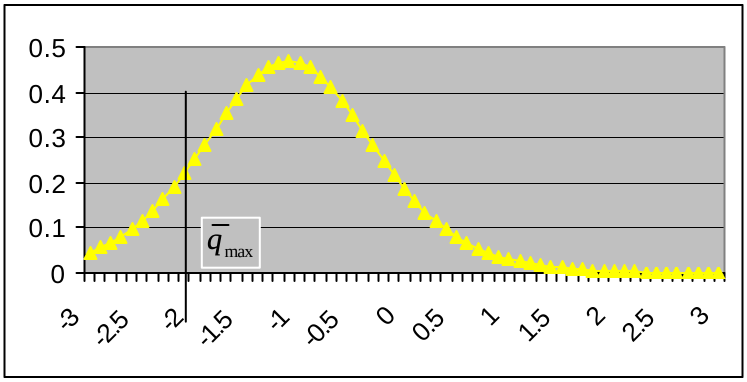

Figure 5 . Example of the function $E_i(q)$ (item AR0031).

Let we assume that in the given test of K items the examinee answer write on items $i_1,\ldots,i_R$ and answer wrong on items $i_{R+1},\ldots,i_K$, where $0 \le R \le K$. Then his/her likelihood will be:

$$L(\overline{u},q) = g(q) \cdot \prod_{k=1}^{R} P_{i_k}(q) \cdot \prod_{k=R+1}^{K} Q_{i_k}(q)$$

Assuming that prior distribution of examinee abilities is normal and using equations (9) – (11) we will have:

$$(L(\overline{u},q))' = L(\overline{u},q) \cdot (-q + R(q))$$

where $R(q) = D \cdot (\sum_{k=1}^{R} a_{i_k} \cdot E_{i_k}(q) + \sum_{k=R+1}^{K} a_{i_k} \cdot e_{i_k}(q))$. Thus problem of maximization likelihood can be presented as solution of equation:

$$q = R(q) . \qquad (12)$$

Proposition 2. If equation (12) has a solution and the function $R(q)$ has continuous derivative and $R'(q) < 1$ for all $q \in [q_{\min}, q_{\max}]$ then this solution is unique.

Proof. If there are two solutions q_1 and q_2 then we can assume $q_2 > q_1$. Due to (12) we have $q_2 - q_1 = R(q_2) - R(q_1) = R'(\tilde{q})(q_2 - q_1) < q_2 - q_1$ which is a contradiction.

As we show before any incorrect answer decreasing derivative of the function $R(q)$ because functions $e_i(q), i = 1,\ldots, M$ are monotone decreasing. In the case of right answer the correspondent function

$E_i(q)$ has rather narrow range of abilities, in which it is increasing. In the point $q = b_i$ and in the the area close to b_i, where the solution of maximum likelihood equation is usually located and where item is used in CAT testing process, the function $E_i(q)$ is monotone decreasing. Thus for usual CAT sequence we expect to found that $R'(q) < 1$, which was confirmed in our simulation studies. Of course, with special construction of a test sequence and items consisting of it, one can reach situation where the relation $R'(q) < 1$ is not holding and equation (12) can have more than one solutions, which was found by F. Samejima (1973), but it is should not happened in the CAT environment .

Proposition 3. If for any feasible area $q \in B(\Theta)$ relation $R'(q) < 1$ is hold and CAT driven by maximum likelihood (instead of Baysian) is not divergent then statement of Proposition 1 or it Corollary is true.

References.

Kantorovich, L. V. (1968). Functional analysis and applied mathematics. Washington, DC: NBS.

Krass, I. A. (1998, April). Application of direct optimization for On-Line calibration in computer adaptive testing. Paper presented at the 1998nd Annual meeting of National Council on Measurement in Education, San Diego, CA.

Levine, M. V. Williams B. A. (1998). Development and Evaluation of Online Calibration Procedures. (Algorithm Design and Measurement Services, Inc. TCN # 96-216) Champaign, IL.

Owen, R. J. (1975). A Bayesian sequential procedure for quantal response in the context of adaptive mental testing. Journal of American Statistical Association, 70, 351-356.

Samejima, F. (1973). A comment on Birnbaum's three-parameter logistic model in the latent trait theory. Psychometrika, 38, 221-233.

Segall, D. O., Moreno, K. E., Bloxom, B. M., & Hetter, R. D. (1997). Psychometric procedures for administering CAT-ASVAB. In W. A. Sands, B. K. Waters, & J. R. McBride (Eds.), Computerized Adaptive Testing (pp. 131-140). Washington, D C: American Psychological Association.

Zimowski, M. F., & Bock, R. D. (1987). Full-information item factor analysis from the ASVAB CAT pool. (Methodology Research Center Report #87-1), Chicago: University of Chicago.